EXCITED STATES FOR THE DIRAC OSCILLATOR VIA WIGNER-HEISENBERG ALGEBRA

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Resume.

In this work we investigate an interesting quantum system, the so-called Dirac oscillator, first introduced by Moshinsky-Szczepaniak [1]; its spectral resolution will be investigated with the help of techniques of super-realization of the Wigner-Heisenberg (WH) algebra [2].

The Wigner-Heisenberg algebra is given by following (anti-)commutation relations

$$H = \frac{1}{2}[a^-, a^+]_+, \quad [H, a^\pm]_- = \pm a^\pm, \quad [a^-, a^+]_- = 1 + \nu R, \quad [R, a^\pm]_+ = 0, \quad R^2 = 1, \quad (1)$$

where ν is a real constant associated to the Wigner parameter [2]. Note that when $\nu = 0$ we have the standard Heisenberg algebra.

Key words. Dirac oscillator; Wigner-Heisenberg algebra; supersymmetry in quantum mechanics.

Introduction.

The Dirac oscillator with different interactions has been treated by Castaños *et al.* [3] and by Dixit *et al.* [4]. These works motivate the construction of a new linear Hamiltonian in terms of the momentum, position and mass coordinates, through a set of seven mutually anticommuting 8x8-matrices yielding a representation of the Clifford algebra $C\ell_7$ [5]. In such an analysis is shown, there is an inherent assymmetry in the spectrum positive and negative energies.

Adding an "anomalous momentum" in the form of a (nonlocal) linear and hermitian interaction, in the (noncovariant) Dirac free particle equation with mass M and spin-$\frac{1}{2}$, in the natural system of units [6],

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}.\vec{p} + M\beta)\psi, \quad (2)$$

one obtains the equation for the Dirac oscillator [1]:

$$i\frac{\partial\psi}{\partial t} = \{\vec{\alpha}.(\vec{p}+\vec{\pi}) + M\beta\}\psi = H_D\psi, \tag{3}$$

where M and ω are, respectively, the mass of the particle and the frequency of the oscillator. With $\vec{\pi} = -iM\omega\beta\vec{r}$.

Methods. In this work we investigate the Dirac oscillator, its spectral resolution has been found with the help of techniques of super-realization of the WH algebra in the context of non-relativist quantum mechanics.

The equation for the Dirac oscillator

$$H_D\psi(\vec{r}) = E\psi(\vec{r}), \quad H_D = \sigma_r\{p_r + \frac{i}{r}(\vec{\sigma}.\vec{L} + \mathbf{1})\}\Sigma_1 + M(\Sigma_3 - \omega r\sigma_r\Sigma_2), \tag{4}$$

where Σ_i and σ_i are two sets of the Pauli matrices so that Σ_i and σ_i commute.

To solve the equation (4), following the usual procedure, we consider the second order differential equation,

$$\tilde{H}_D\psi(\vec{r}) = \tilde{E}\psi(\vec{r}), \quad \tilde{H}_D = \frac{H_D^2 - M^2\mathbf{1}}{2M}, \quad \tilde{E} = \frac{E^2 - M^2}{2M}. \tag{5}$$

Thus, the isotropic 3D SUSY harmonic oscillator with spin-$\frac{1}{2}$, of which some supersymmetric aspects were discussed via the super-realization of the WH algebra. That Hamiltonian is given by [7–9]

$$\begin{aligned}
\tilde{H}_D &= \begin{pmatrix} \tilde{H}_{D1} & 0 \\ 0 & \tilde{H}_{D2} \end{pmatrix} = H_{Ui}, \\
\tilde{H}_{D1} &= \frac{1}{2}\{-\frac{1}{M}(\frac{\partial}{\partial r} + \frac{1}{r})^2 + M\omega^2 r^2 \\
&+ \frac{1}{M}r^{-2}(\vec{\sigma}.\vec{L})(\vec{\sigma}.\vec{L} + 1) - 2(\vec{\sigma}.\vec{L} + \frac{3}{2})\omega\} \\
\tilde{H}_{D2} &= \frac{1}{2}\{-\frac{1}{M}(\frac{\partial}{\partial r})^2 + M\omega^2 r^2 \\
&+\frac{1}{M}r^{-2}(\vec{\sigma}\cdot\vec{L})(\vec{\sigma}\cdot\vec{L} + 1) + 2(\vec{\sigma}\cdot\vec{L} + \frac{3}{2})\omega\}.
\end{aligned} \tag{6}$$

We consider a unitary operator in terms of the radial projection of the spin,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_r \end{bmatrix} = U^{-1} = U^{\dagger}, \tag{7}$$

to obtain the following relation between the transformed Dirac Hamiltonian, $\tilde{H}_D$, the 3D Wigner Hamiltonian, $H(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$, and the SUSY Hamiltonian, $H_{\rm SUSI}$ [10]:

$$H_{\rm SUSI} = U\tilde{H}_D U^\dagger = H(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) - \frac{1}{2}\{1+2(\sigma\cdot\vec{\vec{L}}+\mathbf{1})\Sigma_3\}\omega\Sigma_3. \tag{8}$$

Expressing the right hand side in terms of the 3D Wigner ladder operators of the 3D Wigner oscillator, we obtain the following Hermitian supercharges:

$$\begin{aligned} Q_1 &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & B^+(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) \\ B^-(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) & 0 \end{bmatrix} \\ Q_2 &= \frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -B^+(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) \\ B^-(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) & 0 \end{bmatrix}. \end{aligned} \tag{9}$$

These supercharges satisfy a graded extent of the Lie algebra, $S(2)$,

$$[Q_i, Q_j]_+ = 2\delta_{ij}H_{\rm SUSI}, \quad [H_{\rm SUSI}, Q_i]_- = 0, i = 1, 2, \tag{10}$$

where

$$B^\pm(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) = \frac{1}{\sqrt{2M\omega}}\{\pm(\frac{\partial}{\partial r}+\frac{1}{r}) + \frac{1}{r}(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) - M\omega r\}. \tag{11}$$

Results and Discussion.

We construct the energy eigenvalue and eigenfunctions for the excited states for the Dirac 3D-isotropic oscillator. In this work, we investigate the supersymmetric degeneracy of the 3D Dirac oscillator. The SUSY and the second order Hamiltonians are of same spectrum because both SUSY and transformed Dirac Hamiltonians are related by a unitary transformation. We see that the energy spectra of bosonic and fermionic sectors, in case (i), $j = \ell + \frac{1}{2}$, does not depend of the angular momentum; and in the case (ii), $j = (\ell+1) - \frac{1}{2}$, depend of the angular momentum and hence each sector has a degeneracy $(N\pm 1, j\mp 1), (N\pm 2, j\mp 2), \cdots$. With $N = 2m + \ell, \quad m = 0, 1, 2, 3, ...$

Conclusion.

Therefore, the important connection for the Dirac oscillator with the Wigner-Heisenberg algebra [11–13], satisfying the concomitant general oscillator quantum rule of Wigner, have explicited in this work. Thus, we investigate the Dirac oscillator via second order differential equation. In a forthcoming paper we will investigate the Dirac oscillator in the coherent states.

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